

Random complex zeroes, II. Perturbed lattice

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Abstract

We show that the flat chaotic analytic zero points (i.e. zeroes of a random entire function $\psi(z) = \sum_{k=0}^{\infty} \zeta_k \frac{z^k}{\sqrt{k!}}$ where ζ_0, ζ_1, \dots are independent standard complex-valued Gaussian variables) can be regarded as a random perturbation of a lattice in the plane. The distribution of the distances between the zeroes and the corresponding lattice points is shift-invariant and has a Gaussian-type decay of the tails.

Introduction

We consider the (random) set S of zeroes of a random entire function $\psi: \mathbb{C} \rightarrow \mathbb{C}$,

$$(0.1) \quad \psi(z) = \sum_{k=0}^{\infty} \zeta_k \frac{z^k}{\sqrt{k!}},$$

where ζ_0, ζ_1, \dots are independent standard complex-valued Gaussian random variables; that is, the distribution $\mathcal{N}_{\mathbb{C}}(0, 1)$ of each ζ_k has the density $\pi^{-1} \exp(-|w|^2)$ with respect to the Lebesgue measure m on \mathbb{C} . Well-known as the flat CAZP (‘chaotic analytic zero points’), this model is distinguished by invariance of the distribution of zero points with respect to the motions of the complex plane, see [11] for details and references.

Toy models

It is instructive to compare the flat CAZP with simpler (‘toy’) models of random point processes in the plane, especially, random perturbations of a

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lattice. The first toy model: each point of the lattice $\sqrt{\frac{\pi}{2}}\mathbb{Z}^2 = \{\sqrt{\frac{\pi}{2}}(k + li) : k, l \in \mathbb{Z}\}$ is deleted at random, independently of others, with probability $1/2$; the remaining points are a random set S_1 . For smooth functionals (linear statistics)

$$Z_{L,h}(S) = \sum_{z \in S} h\left(\frac{z}{\sqrt{L}}\right),$$

where $h : \mathbb{C} \rightarrow \mathbb{R}$ is a compactly supported smooth function, mean values are similar for $L \rightarrow \infty$,

$$\begin{aligned}\mathbb{E}Z_{L,h}(S) &\sim L \frac{1}{\pi} \int h \, dm, \\ \mathbb{E}Z_{L,h}(S_1) &\sim L \frac{1}{\pi} \int h \, dm\end{aligned}$$

(here and below, m always stands for the Lebesgue measure), but fluctuations of S_1 are much stronger:

$$\begin{aligned}\text{Var } Z_{L,h}(S) &\sim \frac{\text{const}}{L} \|\Delta h\|^2, \\ \text{Var } Z_{L,h}(S_1) &\sim \text{const} \cdot L \|h\|^2,\end{aligned}$$

see [11], the end of the introduction.

The second toy model: points of the lattice $\sqrt{\pi}\mathbb{Z}^2$ move independently, giving

$$S_2 = \{\sqrt{\pi}(k + li) + \eta_{k,l} : k, l \in \mathbb{Z}\},$$

where $\eta_{k,l}$ are independent standard complex Gaussian random variables. We have [11]

$$\begin{aligned}\mathbb{E}Z_{L,h}(S_2) &\sim L \frac{1}{\pi} \int h \, dm, \\ \text{Var } Z_{L,h}(S_2) &\sim \text{const} \cdot \|\nabla h\|^2;\end{aligned}$$

the latter is closer (than $L\|h\|^2$) to $L^{-1}\|\Delta h\|^2$, but still dissimilar.

Asymptotic similarity to S can be reached (see the third toy model in [11]) by inventing special correlation between perturbations $\eta_{k,l}$.

Main result

Discarding toy models and asymptotic properties, we come to the idea of CAZP as a perturbed lattice,¹

$$S = \{\sqrt{\pi}(k + li) + \xi_{k,l} : k, l \in \mathbb{Z}\}$$

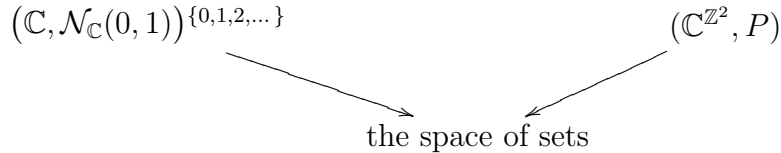
¹Area of its cells must equal π . Any lattice with this cell area may be used.

for some (dependent) complex-valued random variables $\xi_{k,l}$. Of course, it can happen that all points of S are far from the origin, in which case $|\xi_{0,0}|$ is necessarily large. However, that is an event of small probability. We may hope for fast decay of the probability $\mathbb{P}(|\xi_{k,l}| \geq r)$ for large r , uniformly in k, l . The uniformity becomes trivial if random variables $\xi_{k,l}$ are identically distributed. Taking into account invariance of CAZP under shifts of \mathbb{C} we may hope for invariance of $(\xi_{k,l})$ under lattice shifts. The hopes come true, which is our main result, formulated below. Random variables are treated as measurable functions on the space Ω of two-dimensional arrays $\xi: \mathbb{Z}^2 \rightarrow \mathbb{C}$ of complex numbers.

Main Theorem. *There exists a probability measure P on (the Borel σ -field of) the space $\Omega = \mathbb{C}^{\mathbb{Z}^2}$, invariant under shifts of \mathbb{Z}^2 and such that*

- (a) *the random set $\{\sqrt{\pi}(k + li) + \xi_{k,l}: k, l \in \mathbb{Z}\}$ is distributed like the flat CAZP;*
- (b) $\mathbb{E} \exp(\varepsilon |\xi_{0,0}|^2) < \infty$ *for some $\varepsilon > 0$.*

Item (a) needs some comments. A ‘random set’ is a measurable map from Ω to a space of sets. We need only locally finite subsets of \mathbb{C} . The Borel σ -field on that space is generated by functions $S \mapsto \sum_{z \in S} h(z)$, where h runs over compactly supported continuous (or just Borel) functions $\mathbb{C} \rightarrow \mathbb{R}$. Alternatively, we may represent each set S by its counting measure, which is the same, since the random entire function ψ has only simple zeroes (almost surely). Item (a) means that the two maps



induce the same measure on the space of sets. Here the first map sends a sequence of coefficients ζ_0, ζ_1, \dots into the set of zeroes of $\psi(z) = \sum \zeta_k z^k / \sqrt{k!}$, while the second map sends an array $(\xi_{k,l})_{k,l}$ into the set $\{\sqrt{\pi}(k + li) + \xi_{k,l}: k, l \in \mathbb{Z}\}$. The latter set is locally finite (almost surely), which is a part of item (a).

The second map on the diagram intertwines natural (measure preserving) actions of lattice shifts on Ω and the space of sets. For the first map, the situation is more complicated; only a *projective* action of shifts is naturally defined on the space of entire functions (or their coefficients), see [11].

Our construction of the matching between the flat CAZP and the lattice points $\sqrt{\pi}\mathbb{Z}^2$ is not explicit. For this reason, the main theorem gives no information about correlations between $\xi_{k,l}$. On large distances, the correlation

function of the underlying Gaussian process decays rapidly (see for example [11, Sect. 3.2]). Probably, $\xi_{k,l}$ can be chosen as to be nearly independent on large distances. It is also compatible with the result of [12]. On small distances we expect a negative correlation, for two reasons: the well-known repulsion of close zeroes [5, 3], and the center-of-mass conservation discussed in [11, Introduction].

Explicit matching?

It could be very useful to find an explicit matching between the flat CAZP and the lattice points, or equivalently, a transportation of the Lebesgue measure $\frac{1}{\pi}m$ to the counting measure n_ψ of CAZP. By ‘transportation’ we mean a map $T: \mathbb{C} \rightarrow \mathbb{C}$ such that $\pi n_\psi = T_*m$. Of course, we are interested in stationary random transportations T with fast decay of the tails of the distribution of $Tz - z$. Here, we suggest a natural and explicit construction of the transportation which, in our opinion, deserves a better look²

Consider the gradient field of the stationary random potential [11, Introduction]

$$(0.2) \quad \varphi(z) = \frac{1}{2} \log |\psi(z)| - \frac{1}{4} |z|^2$$

(additional factor $\frac{1}{2}$ on the RHS will be convenient later); the distributional Laplacian of φ equals $\pi n_\psi - m$. The only local minima of the function φ are the points where it equals $-\infty$ (since φ is superharmonic everywhere except of these points). We say that the point w belongs to the *basin* of a zero point $z \in \psi^{-1}(0)$ if $\nabla\varphi(w) \neq 0$, and the gradient trajectory passing through w terminates at z . In other words, if we put a ball at the point $(w, \varphi(w))$ on the graph of the function φ , then under the gravitation force (and without inertia) the ball will fall through at the point $(z, -\infty)$.

We expect that almost surely one obtains a cellulation of the plane on *finite cells*, each of them being the basin of some random zero point; i.e. with probability one, nothing escapes to infinity, and nothing arrives from infinity. On the boundary of each cell, the gradient $\nabla\varphi$ has zero normal component. Since each cell contains exactly one random zero point, by Green’s theorem applied to the function φ , the area of each cell must equal π . Define T as the map that sends each basin into the corresponding zero point, then T transports the measure $\frac{1}{\pi}m$ to the measure n_ψ . It would be interesting to obtain a good estimate for the diameters of the basins.

²It is inspired by the celebrated Moser homotopic construction [9, Sect. 4] of the diffeomorphism that transports one volume measure to another.

Other point processes

It is instructive to think about possible counterparts of the main theorem for simpler random sets such as the first toy model (Bernoulli process) or the Poisson point process. Here, $\mathbb{E}|\xi_{0,0}|$ must be infinite, since for a finite fragment of size $n \times n$ the transportation cost between the random set and the lattice, divided by the number of points, typically exceeds $\text{const} \cdot \sqrt{\ln n}$ [2, 14]. The large gap between $\mathbb{E}|\xi_{0,0}|$ and $\mathbb{E} \exp(\varepsilon |\xi_{0,0}|^2)$ shows that the random zeroes are distributed much more evenly than independent random points (see [10], Fig. 1 and comments to it). Note also that (i) stationary matchings between random and deterministic sets are closely related with so-called extra head schemes [7]; (ii) existence of a stationary matching between the Poisson point process on \mathbb{R}^2 and the lattice follows from existence of a ‘stable marriage of Poisson and Lebesgue’ announced recently [7, 6].

The reader’s guide, I (informal)

The proof of the main theorem is based on the formula

$$(0.3) \quad 2\pi \, dn_\psi = \Delta \log |\psi| \, dm,$$

where n_ψ is the counting measure on the set of zeroes of ψ . The desired array $(\xi_{k,l})_{k,l}$ may be thought of as a bijective correspondence (‘marriage’) between zeroes of ψ and lattice points. The distance $|\xi_{k,l}|$ between corresponding points (‘fiancé’ and ‘bride’) must be controlled as to ensure item (b) of the theorem. It is instructive to try first a simpler condition, say, $|\xi_{k,l}| \leq 100$ for all k, l . (In fact, it is too much for a typical ψ , but let us try to prove it anyway.) If such $\xi_{k,l}$ exist, then clearly

$$(0.4) \quad n_\psi(U) \leq n(U_{+r}) \quad \text{and} \quad n(U) \leq n_\psi(U_{+r})$$

for every $U \subset \mathbb{C}$; here U_{+r} stands for the r -neighborhood of U , $r = 100$, and n is the counting measure on the lattice. (No measurability is required of U , since the measures n, n_ψ are discrete. However, it does not harm to assume U to be a bounded domain with a smooth boundary.) In fact, (0.4) is necessary *and sufficient*, which is basically the well-known ‘marriage lemma’. Using (0.4) as a sufficient condition, we may replace n by $\frac{1}{\pi}m$ at the expense of some change of the constant r .

Taking into account that $(\mathbb{E}|\psi(z)|^2)^{1/2} = \exp(\frac{1}{2}|z|^2)$ one could expect naïvely that the ‘potential’ (0.2) is bounded on \mathbb{C} . This can be used to show that $\int_U \Delta \varphi \, dm \leq m(U_{+r} \setminus U)$ and $-\int_{U_{+r}} \Delta \varphi \, dm \leq m(U_{+r} \setminus U)$, which gives (0.4) since $\Delta \varphi = \pi n_\psi - m$. Singularity of the potential at zeroes is not

an obstacle, since we can replace n_ψ by its convolution with a compactly supported smooth measure.

The argument sketched above does not work since the smoothed random potential is unbounded (for almost all ψ). However, it can be mended. Rare fluctuations appear somewhere on the infinite plane \mathbb{C} , probably far from the origin. In order to get item (b) of the theorem we need some locality; $|\xi_{0,0}|$ should not be large, whenever the potential is not large in an appropriate neighborhood of the origin. Restrictions on $|\xi_{k,l}|$ should be adaptive, they should be relaxed around large values of the potential. This idea is formalized by introducing on \mathbb{C} a metric ρ that depends on ψ , and considering ρ -neighborhoods U_{+r} . Such metrics ρ are not shift-invariant; rather, the probability distribution on the space of these metrics is shift-invariant. Finally, shift invariance of P is ensured.

The reader's guide, II (more formal)

To build the metric ρ , we use an idea borrowed from [8, Section 1.4]. A $\text{Lip}(1)$ -function $R: \mathbb{R}^d \rightarrow (0, \infty)$, $|R(x) - R(y)| \leq |x - y|$, gives rise to the metric

$$\rho(x, y) = \inf_{\gamma} \int_{\gamma} \frac{|dz|}{R(z)}$$

where the infimum is taken over all piece-wise C^1 -curves γ in \mathbb{R}^d connecting the points x and y . We shall call ρ a *special metric* on \mathbb{R}^d , corresponding to R (in which case R is always assumed to be a positive $\text{Lip}(1)$ function). In Sect. 1, we list the properties of the metric ρ needed in the rest of the paper; in Sect. 2, we construct a Whitney-type partition of unity subordinated to the metric ρ .

This partition of unity is needed for the following potential theory lemma which lies in the heart of our argument:

Main Lemma. *There exists const such that if a C^2 -function $u: \mathbb{R}^d \rightarrow \mathbb{R}$ and a special metric ρ (corresponding to R) satisfy $|u(x)| \leq \text{const} \cdot R^2(x)$ and $\Delta u(x) \geq -1$ for all x , then*

$$\int_U \Delta u \, dm \leq m(U_{+4} \setminus U), \quad - \int_{U_{+4}} \Delta u \, dm \leq m(U_{+4} \setminus U)$$

for every compact set $U \subset \mathbb{R}^d$.

Here $U_{+4} = \{y: \exists x \in U \, \rho(x, y) \leq 4\}$. For the proof see Sect. 3.

To apply this lemma, we smooth the random potential φ (see (0.2)) setting $u = \varphi * \chi$, where χ is a compactly supported smooth function

$\chi: \mathbb{C} \rightarrow [0, \infty)$ such that $\int \chi dm = 1$ and $\chi(-z) = \chi(z)$ for all z (the choice of χ influences only constants), and define the Lip(1)-function R as follows (see (4.1)):

$$R(z) = \max_w \left(\sqrt{\text{Const} \cdot (1 + (|\varphi| * \chi)(w))} - |w - z| \right).$$

Then the main lemma combined with the marriage lemma and the general inequality $|x - y| \leq 2^{3\rho(x,y)} R(x)$ (see Lemma 1.4) give us a matching between CAZP and the lattice points $\sqrt{\pi}\mathbb{Z}^2$ such that for every matched pair $z \in \psi^{-1}(0)$ and $\sqrt{\pi}(k + il)$

$$|z - \sqrt{\pi}(k + il)| \leq R(\sqrt{\pi}(k + il))$$

(see Theorem 4.3).

It will be shown (see Lemma 5.1) that $\mathbb{E} \exp(cR^2(x)) \leq C$. This will give us the subgaussian decay of the tails, thus proving Item (b) of the main theorem.

It worth mentioning that the Gaussian nature of the random function ψ is used only once, in Sect. 5, when checking the inequality $\mathbb{E} \exp(\text{const} \cdot |\varphi(z)|) \leq \text{Const}$ (uniformly in z). For every random entire function satisfying the inequality, its zeroes are a ‘perturbed lattice’ satisfying $\mathbb{E} \exp(\text{const} \cdot |\xi_{k,l}|^2) < \text{Const}$ (uniformly in k, l). On the other hand, shift-invariance of $(\xi_{k,l})_{k,l}$ is achieved via shift-invariance of zeroes of ψ ; the latter was verified using the Gaussian distribution.

Convention

Most of the steps in the proof of the main theorem do not use any special properties of the complex plane and will be done in the Euclidean space \mathbb{R}^d . Throughout, ‘Const’ and ‘const’ mean positive constants (sufficiently large and sufficiently small, respectively) depending on the dimension d only, the values of these constants can be changed at each occurrence. By $B(x; r)$ we always denote the closed ball $\{y: |x - y| \leq r\}$.

Acknowledgment

We thank Michael Krivelevich for providing us with the reference [1] and its discussion, Fëdor Nazarov for bringing to our attention the relevance of the classical Whitney construction, and Yuval Peres for telling us about results of the papers [2, 14, 6, 7].

1 A class of metrics in \mathbb{R}^d

Suppose ρ is the special metric corresponding to a Lip(1)-function $R: \mathbb{R}^d \rightarrow (0, \infty)$; i.e.

$$\rho(x, y) = \inf_{\gamma} \int_{\gamma} \frac{|dz|}{R(z)}$$

where the infimum is taken over all piece-wise C^1 -curves γ in \mathbb{R}^d connecting the points x and y . We prove several simple facts about the special metric ρ .

1.1 Lemma. *For all $x, y \in \mathbb{R}^d$,*

$$|y - x| \leq \frac{1}{2}R(x) \quad \text{implies} \quad \frac{1}{2}R(x) \leq R(y) \leq \frac{3}{2}R(x).$$

Proof. $R(y) \leq R(x) + |x - y| \leq \frac{3}{2}R(x)$, and $R(x) \leq R(y) + |x - y| \leq R(y) + \frac{1}{2}R(x)$ which gives $\frac{1}{2}R(x) \leq R(y)$. \square

Given a piece-wise C^1 -curve γ which starts at x and terminates at y , we define the *index* $N(\gamma) \in \mathbb{N}$ as the length N of the chain of points $x_0 = x, x_1, \dots, x_N = y$ on γ constructed one after another as follows. Having x_j , we consider the rest $[x_j, y]_{\gamma}$ of the curve γ (the part of γ which starts at x_j and terminates at y). If $[x_j, y]_{\gamma} \subset B(x_j; \frac{1}{2}R(x_j))$ then the process stops at $x_{j+1} = y$, $N = j + 1$. Otherwise x_{j+1} is the first point on $[x_j, y]_{\gamma}$ lying on the sphere $\partial B(x_j, \frac{1}{2}R(x_j))$, and the process is continued.

1.2 Lemma. $N(\gamma) \leq 3 \int_{\gamma} \frac{|dz|}{R(z)} + 1$.

Proof. Denote by $[x_j, x_{j+1}]_{\gamma}$ the part of γ between x_j and x_{j+1} . If $z \in [x_j, x_{j+1}]_{\gamma}$, then $R(z) \leq \frac{3}{2}R(x_j)$ by Lemma 1.1, therefore, denoting $N = N(\gamma)$ and assuming $N > 1$ (otherwise there is nothing to prove),

$$\begin{aligned} \int_{\gamma} \frac{|dz|}{R(z)} &= \sum_{j=0}^{N-1} \int_{[x_j, x_{j+1}]_{\gamma}} \frac{|dz|}{R(z)} \geq \sum_{j=0}^{N-2} \int \dots \\ &\geq \sum_{j=0}^{N-2} \frac{2}{3R(x_j)} \cdot |x_j - x_{j+1}| = \sum_{j=0}^{N-2} \frac{2}{3R(x_j)} \cdot \frac{1}{2}R(x_j) = \frac{N-1}{3}. \end{aligned}$$

\square

1.3 Lemma. *Let ρ be a special metric on \mathbb{R}^d , corresponding to R . Then*

$$|x - y| \leq \frac{1}{2}R(x) \quad \text{implies} \quad \rho(x, y) \leq 1.$$

Proof. Let $[x, y] \subset \mathbb{R}^d$ be the straight segment with end-points at x and y . Then

$$\rho(x, y) \leq \int_{[x, y]} \frac{|dz|}{R(z)} \stackrel{1.1}{\leq} \frac{|x - y|}{\frac{1}{2}R(x)} \leq 1.$$

□

1.4 Lemma. *Let ρ be a special metric on \mathbb{R}^d , corresponding to R . Then*

$$|x - y| \leq 2^{3\rho(x, y)} R(x).$$

Proof. Given $\varepsilon > 0$, choose a curve γ connecting the points x and y , such that $\int_{\gamma} \frac{|dz|}{R(z)} < \rho(x, y) + \varepsilon$. Let $x_0 = x, x_1, \dots, x_N = y$ be a partition of γ constructed above, $N = N(\gamma)$. Then $R(x_j) \leq (\frac{3}{2})^j R(x) \leq 2^j R(x)$, and

$$\begin{aligned} |x - y| &\leq \sum_{j=0}^{N-1} |x_j - x_{j+1}| \leq \frac{1}{2} \sum_{j=0}^{N-1} R(x_j) \\ &\leq \frac{1}{2} R(x) \sum_{j=0}^{N-1} 2^j < 2^{N-1} R(x) \stackrel{1.2}{<} 2^{3(\rho(x, y) + \varepsilon)} R(x). \end{aligned}$$

□

2 Whitney-type partitions of unity

For a smooth function $f: \mathbb{R}^d \rightarrow \mathbb{R}$ we denote by $|\nabla f(x)|$ a norm of the gradient vector, say, $|\nabla f(x)| = \sum_k \left| \frac{\partial}{\partial x_k} f(x_1, \dots, x_d) \right|$ (the choice of the norm does not matter), and by $|\nabla^2 f(x)|$ a norm of the matrix of second derivatives, say, $|\nabla^2 f(x)| = \sum_{k, l} \left| \frac{\partial^2}{\partial x_k \partial x_l} f(x_1, \dots, x_d) \right|$.

2.1 Theorem. *Let ρ be a special metric corresponding to R , and $U \subset \mathbb{R}^d$ be a closed set. Then there exist a C^2 -function $f: \mathbb{R}^d \rightarrow [0, 1]$ and Const such that*

$$\begin{aligned} f(x) &= 1 \quad \text{for all } x \in U, \\ f(x) &= 0 \quad \text{for all } x \in \mathbb{R}^d \setminus U_{+4}, \\ \int (R|\nabla f| + R^2|\nabla^2 f|) dm &\leq \text{Const} \cdot \int_{U_{+4} \setminus U} f dm. \end{aligned}$$

Here and henceforth $U_{+r} = \{y: \exists x \in U \rho(x, y) \leq r\}$ is the r -neighborhood of U with respect to ρ .

We denote for convenience $B(x) = B(x; \frac{1}{2}R(x))$ and $\frac{1}{2}B(x) = B(x; \frac{1}{4}R(x))$. By Lemma 1.3,

$$(2.2) \quad \forall x \quad \forall y, z \in B(x) \quad \rho(y, z) \leq 2.$$

The following fact follows immediately from [8, Lemma 1.4.9].

2.3 Lemma. *There exist a countable locally finite set $S \subset \mathbb{R}^d$ and Const such that*

- (a) *the balls $\{\frac{1}{2}B(s) : s \in S\}$ cover \mathbb{R}^d ,*
- (b) *the multiplicity of the covering by twice larger balls $\{B(s) : s \in S\}$ does not exceed Const .*

The next lemma is essentially Theorem 1.4.10 from [8].

2.4 Lemma. *There exist constants const , Const , and C^2 -functions $f_s : \mathbb{R}^d \rightarrow [0, 1]$ for $s \in S$ (where S is given by Lemma 2.3) such that*

- (a) *$f_s(x) = 0$, unless $x \in B(s)$;*
- (b) *for all $x \in \mathbb{R}^d$,*

$$\sum_{s \in S} f_s(x) = 1;$$

- (c) *for all $s \in S$,*

$$\int f_s dm \geq \text{const} \cdot R^d(s);$$

- (d) *for all $s \in S$,*

$$\sup_{x \in \mathbb{R}^d} |\nabla f_s(x)| \leq \frac{\text{Const}}{R(s)}, \quad \sup_{x \in \mathbb{R}^d} |\nabla^2 f_s(x)| \leq \frac{\text{Const}}{R^2(s)}.$$

Proof. We start with smooth functions $g_s : \mathbb{R}^d \rightarrow [0, 1]$ that satisfy (a, d) and $g_s(x) = 1$ whenever $x \in \frac{1}{2}B(s)$; the latter implies (c) (for g_s). Their sum

$$g = \sum_{s \in S} g_s$$

satisfies, for all $x \in \mathbb{R}^d$,

$$\text{const} \leq g(x) \leq \text{Const}.$$

Indeed, the multiplicity of the covering (see Lemma 2.3) is an upper bound; the lower bound (just 1) follows from 2.3(a).

It follows from condition (d) (for g_s) that

$$|\nabla g(x)| \leq \frac{\text{Const}}{R(x)}, \quad |\nabla^2 g(x)| \leq \frac{\text{Const}}{R^2(x)}$$

for all x . It remains to take $f_s = g_s/g$. □

Proof of Theorem 2.1. Lemmas 2.3, 2.4 give us S and $(f_s)_{s \in S}$; we construct

$$f = \sum_{s: B(s) \subset U_{+4}} f_s.$$

By 2.4(a), $f(x) = 0$ for all $x \notin U_{+4}$. By 2.4(b), $f(x) = 1$ for all $x \in U$ and moreover, for all $x \in U_{+2}$, since by (2.2), $B(s) \cap U_{+2} \neq \emptyset$ implies $B(s) \subset U_{+4}$.

We introduce a seminorm $\|\cdot\|$,

$$\|g\| = \int (R|\nabla g| + R^2|\nabla^2 g|) dm$$

for smooth functions $g: \mathbb{R}^d \rightarrow \mathbb{R}$ such that this integral converges. By 2.4(a,c,d),

$$\|f_s\| \leq \text{Const} \cdot \int f_s dm$$

for all $s \in S$. Clearly, $\|f\| \leq \sum_{s: B(s) \subset U_{+4}} \|f_s\|$, but moreover,

$$\|f\| \leq \sum_{s: B(s) \subset U_{+4} \setminus U} \|f_s\|,$$

since, taking into account that $\nabla f = 0$ outside $U_{+4} \setminus U_{+2}$, we have

$$\begin{aligned} \|f\| &= \int_{U_{+4} \setminus U_{+2}} (R|\nabla f| + R^2|\nabla^2 f|) dm \\ &\leq \sum_{s: B(s) \subset U_{+4}} \int_{U_{+4} \setminus U_{+2}} (R|\nabla f_s| + R^2|\nabla^2 f_s|) dm, \end{aligned}$$

and the last integral vanishes for s such that $B(s) \cap U \neq \emptyset$. Finally,

$$\|f\| \leq \sum_{s: B(s) \subset U_{+4} \setminus U} \|f_s\| \leq \text{Const} \cdot \sum_{s: B(s) \subset U_{+4} \setminus U} \int f_s dm \leq \text{Const} \cdot \int_{U_{+4} \setminus U} f dm.$$

□

3 The main lemma

3.1 Main Lemma. *There exists const such that if a C^2 -function $u: \mathbb{R}^d \rightarrow \mathbb{R}$ and a special metric ρ (corresponding to R) satisfy*

$$(3.2) \quad |u(x)| \leq \text{const} \cdot R^2(x),$$

$$(3.3) \quad \Delta u(x) \geq -1$$

for all x , then

$$\int_U \Delta u \, dm \leq m(U_{+4} \setminus U), \quad - \int_{U_{+4}} \Delta u \, dm \leq m(U_{+4} \setminus U)$$

for every compact set $U \subset \mathbb{R}^d$.

Still, $U_{+4} = \{y: \exists x \in U \, \rho(x, y) \leq 4\}$.

Proof. Theorem 2.1 gives us a function $f: \mathbb{R}^d \rightarrow [0, 1]$ that equals 1 on U , 0 outside U_{+4} , and satisfies $\int R^2 |\Delta f| \, dm \leq \text{Const} \cdot \int_{U_{+4} \setminus U} f \, dm$. We have

$$\begin{aligned} - \int_{U_{+4}} \Delta u \, dm &= - \int f \Delta u \, dm - \int_{U_{+4} \setminus U} (1 - f) \Delta u \, dm; \\ - \int f \Delta u \, dm &= - \int u \Delta f \, dm \leq \int |u| |\Delta f| \, dm \\ &\leq \text{const} \cdot \int R^2 |\Delta f| \, dm \leq \underbrace{\text{const} \cdot \text{Const}}_{\leq 1} \cdot \int_{U_{+4} \setminus U} f \, dm; \end{aligned}$$

$$\int_{U_{+4} \setminus U} (1 - f) (-\Delta u) \, dm \leq \int_{U_{+4} \setminus U} (1 - f) \, dm;$$

thus,

$$- \int_{U_{+4}} \Delta u \, dm \leq \int_{U_{+4} \setminus U} f \, dm + \int_{U_{+4} \setminus U} (1 - f) \, dm = m(U_{+4} \setminus U).$$

In order to prove the other inequality, we apply Theorem 2.1 to the closed set $\mathbb{R}^d \setminus U_{+4}$ in place of U and take $1 - f$ in place of f . This gives us $f: \mathbb{R}^d \rightarrow [0, 1]$ that equals 1 on U , 0 outside U_{+4} , and satisfies $\int R^2 |\Delta f| \, dm \leq \text{Const} \cdot \int_{U_{+4} \setminus U} (1 - f) \, dm$. We have

$$\begin{aligned} \int_U \Delta u \, dm &= \int f \Delta u \, dm - \int_{U_{+4} \setminus U} f \Delta u \, dm; \\ \int f \Delta u \, dm &= \int u \Delta f \, dm \leq \int |u| |\Delta f| \, dm \\ &\leq \text{const} \cdot \int R^2 |\Delta f| \, dm \leq \int_{U_{+4} \setminus U} (1 - f) \, dm; \\ \int_{U_{+4} \setminus U} f \cdot (-\Delta u) \, dm &\leq \int_{U_{+4} \setminus U} f \, dm; \end{aligned}$$

thus,

$$\int_U \Delta u \, dm \leq \int_{U_{+4} \setminus U} (1 - f) \, dm + \int_{U_{+4} \setminus U} f \, dm = m(U_{+4} \setminus U).$$

□

4 Tying zeroes of an entire function to the lattice points

We fix once and forever a compactly supported smooth function $\chi: \mathbb{C} \rightarrow [0, \infty)$ such that $\int \chi dm = 1$ and $\chi(-z) = \chi(z)$ for all z . The choice of χ influences only constants. Given an entire function $\psi: \mathbb{C} \rightarrow \mathbb{C}$ and a constant (Const), we define a function $R: \mathbb{C} \rightarrow [1, \infty]$ by

$$(4.1) \quad R(z) = \max_w \left(\sqrt{\text{Const} \cdot (1 + (|\varphi| * \chi)(w))} - |w - z| \right);$$

here, as before,

$$(4.2) \quad \varphi(z) = \frac{1}{2} \log |\psi(z)| - \frac{1}{4} |z|^2,$$

$|\varphi|$ is the pointwise absolute value of φ , and $|\varphi| * \chi$ is the convolution of these two functions.

Clearly, $R(z) \geq \text{Const}^{1/2} > 0$ for all z (this time, ‘Const’ is the same as in (4.1)), and the function R is a Lip(1)-function since it is an upper envelope of Lip(1)-functions. Therefore we may construct a special metric ρ corresponding to R . Note that $\rho(x, y) \leq \text{Const}^{-1/2} |x - y|$ for all x, y . If we replace ‘Const’ with ‘4 Const’, we get another function R_2 such that $R_2(z) \geq 2R(z)$ for all z , and another special metric ρ_2 such that $\rho_2(x, y) \leq \frac{1}{2}\rho(x, y)$ for all x, y .

Due to the Lipschitz property, the function R is finite everywhere in \mathbb{C} provided that $R(0) < \infty$.

4.3 Theorem. *There exists Const in (4.1) with the following property. For every entire function ψ satisfying $R(0) < \infty$ there exists a bijection between the lattice $\sqrt{\pi}\mathbb{Z}^2$ and the zero set $\psi^{-1}(0)$ (counting with multiplicities) such that for every pair of corresponding points $z \in \psi^{-1}(0)$, $\sqrt{\pi}(k + li) \in \sqrt{\pi}\mathbb{Z}^2$*

$$(4.4) \quad |z - \sqrt{\pi}(k + li)| \leq R(\sqrt{\pi}(k + li)).$$

Proof. In order to get (4.4) we will show that

$$(4.5) \quad \rho(z, \sqrt{\pi}(k + li)) \leq 5,$$

where ρ is the special metric corresponding to R . In combination with Lemma 1.4 it implies $|z - \sqrt{\pi}(k + li)| \leq 2^{15} R(\sqrt{\pi}(k + li))$; the constant 2^{15} need not appear in (4.4), since it can be absorbed by Const in (4.1). Of course, there is nothing sacred in the constant ‘5’ on the RHS of (4.5); any other constant does the job as well.

Existence of a bijection between the lattice points and the set of zeroes satisfying (4.5) follows from inequalities (to be proven)

$$(4.6) \quad n_\psi(U) \leq n(U_{+5}), \quad n(U) \leq n_\psi(U_{+5})$$

for every compact $U \subset \mathbb{C}$; here $U_{+5} = \{y: \exists x \in U \rho(x, y) \leq 5\}$, n_ψ is the counting measure on the set of zeroes, and n is the counting measure on the lattice. Indeed, we say that the marriage between the bride $\sqrt{\pi}(k+li)$ and the fiancé $z \in \psi^{-1}(0)$ is possible if $\rho(z, \sqrt{\pi}(k+li)) \leq 5$. By the classical marriage lemma (we use its extension due to M. Hall [1, p.7]), the first inequality in (4.6) guarantees that each bride can find a fiancé, and the second inequality in (4.6) yields that each fiancé can find a bride. Then a version of the Cantor-F. Bernstein theorem [1, Theorem 1.1] gives a matching which covers every bride and every fiancé.

We split (4.6) into

$$(4.7) \quad \pi n(U) \leq m(U_{+1}), \quad m(U) \leq \pi n(U_{+1});$$

$$(4.8) \quad \pi n_\psi(U) \leq m(U_{+4}), \quad m(U) \leq \pi n_\psi(U_{+4});$$

here m is the Lebesgue measure on \mathbb{C} . Clearly, (4.7) and (4.8) together imply (4.6).

Inequalities (4.7) are easy to check, taking into account that by Lemma 1.3 the ρ -neighborhood U_{+1} of U contains a sufficiently large Euclidean neighborhood of U , provided that Const in (4.1) is large enough. It remains to prove (4.8).

We will prove a seemingly weaker (but ultimately equivalent) statement:

$$(4.9) \quad \pi n_\psi(V) \leq m(W), \quad m(V) \leq \pi n_\psi(W)$$

whenever V, W are such that $V \subset U \subset U_{+4} \subset W$, $\mathbb{1}_U * \chi = 1$ on V , $\mathbb{1}_{U_{+4}} * \chi = 0$ outside W . (Here $\mathbb{1}_U$ stands for the indicator function of U .) That is sufficient: having (4.9) we get (4.8) after replacing ‘Const’ with ‘4 Const’ in (4.1), as follows. Considering the corresponding R_2, ρ_2 and denoting the r -neighborhood w.r.t. ρ_2 by U^{+r} we have $U^{+r} \supset U_{+2r}$ (since $\rho_2 \leq \frac{1}{2}\rho$). Given \tilde{U} , we may apply (4.9) to $V = \tilde{U}$, $U = V_{+2}$, $W = V^{+4} \supset V_{+8}$ provided that Const is large enough. We get $\pi n_\psi(\tilde{U}) \leq m(\tilde{U}^{+4})$, $m(\tilde{U}) \leq \pi n_\psi(\tilde{U}^{+4})$, which is (4.8). It remains to prove (4.9).

Main lemma 3.1 can be applied to $u = \varphi * \chi$, since condition (3.2) follows from (4.1), and (3.3) is checked readily:

$$\Delta(\varphi * \chi) = (\Delta\varphi) * \chi = (\pi n_\psi - m) * \chi = \pi n_\psi * \chi - m \geq -m.$$

We get

$$\begin{aligned} \int (\mathbb{1}_U * \chi) \Delta \varphi \, dm &= \int_U \Delta(\varphi * \chi) \, dm \leq m(U_{+4} \setminus U), \\ - \int (\mathbb{1}_{U_{+4}} * \chi) \Delta \varphi \, dm &= - \int_{U_{+4}} \Delta(\varphi * \chi) \, dm \leq m(U_{+4} \setminus U). \end{aligned}$$

However, $\int (\mathbb{1}_U * \chi) \Delta \varphi \, dm = \int (\mathbb{1}_U * \chi) \, d(\pi n_\psi - m) = \pi \int (\mathbb{1}_U * \chi) \, dn_\psi - m(U)$, therefore

$$\pi \int (\mathbb{1}_U * \chi) \, dn_\psi \leq m(U_{+4}), \quad m(U) \leq \pi \int (\mathbb{1}_{U_{+4}} * \chi) \, dn_\psi,$$

and we get (4.9):

$$\begin{aligned} \pi n_\psi(V) &\leq \pi \int (\mathbb{1}_U * \chi) \, dn_\psi \leq m(U_{+4}) \leq m(W), \\ m(V) &\leq m(U) \leq \pi \int (\mathbb{1}_{U_{+4}} * \chi) \, dn_\psi \leq \pi n_\psi(W). \end{aligned}$$

The proof of Theorem 4.3 is completed. \square

5 Probabilistic arguments

In the following lemma, (Ω, \mathcal{F}, P) is a probability space.

5.1 Lemma. *Let a random process $\eta: \mathbb{R}^d \times \Omega \rightarrow [0, \infty)$ satisfy*

$$(5.2) \quad \mathbb{E} \exp(c\eta(x)) \leq C$$

for some $C, c \in (0, \infty)$ and all $x \in \mathbb{R}^d$, and let $R: \mathbb{R}^d \times \Omega \rightarrow [0, \infty)$ be a random process defined by

$$R(x) = \max_y \left(\sqrt{\text{Const} \cdot (1 + (\eta * \chi)(y))} - |y - x| \right).$$

Then there are constants c_1 and C_1 such that for all $x \in \mathbb{R}^d$,

$$\mathbb{E} \exp(c_1 R^2(x)) \leq C_1.$$

The constants c_1 and C_1 depend on c, C, Const , the dimension d , and the function χ (introduced in Sect. 4).

Proof. We will prove that

$$(5.3) \quad \mathbb{P} (R(0) > \lambda) \leq C_1 \exp(-c_1 \lambda^2)$$

for all λ large enough. It evidently implies $\mathbb{E} \exp(c_1 R^2(0)) \leq C_1$ (with different c_1 and C_1). For other x , $R(x)$ is treated similarly.

We have

$$\mathbb{P} (R(0) > \lambda) = \mathbb{P} (\exists x (\eta * \chi)(x) > \text{const} \cdot (\lambda + |x|)^2 - 1) .$$

For large λ we may discard ‘ -1 ’ on the RHS (at the expense of changing the constant);

$$\mathbb{P} (R(0) > \lambda) \leq \mathbb{P} (\exists x (\eta * \chi)(x) > \text{const} \cdot \lambda^2 + \text{const} \cdot |x|^2) .$$

Since the function η is non-negative, all values of the convolution $\eta * \chi$ are bounded by its values on a lattice: given χ there are constants c_2 and C_2 such that

$$(\eta * \chi)(x) \leq C_2 \max\{(\eta * \chi)(y) : y \in c_2 \mathbb{Z}^d, |y - x| \leq C_2\} .$$

It follows that

$$\begin{aligned} \mathbb{P} (R(0) > \lambda) &\leq \mathbb{P} (\exists x \in c_2 \mathbb{Z}^d (\eta * \chi)(x) > c_3 \lambda^2 + c_3 |x|^2) \\ &\leq \sum_{x \in c_2 \mathbb{Z}^d} \mathbb{P} ((\eta * \chi)(x) > c_3 \lambda^2 + c_3 |x|^2) . \end{aligned}$$

Let c be a constant from (5.2). Then

$$\mathbb{P} ((\eta * \chi)(x) > c_3 \lambda^2 + c_3 |x|^2) \leq \frac{\mathbb{E} \exp(c(\eta * \chi)(x))}{\exp(c_4 \lambda^2 + c_4 |x|^2)}$$

with $c_4 = c_3 c$. However,

$$\exp(c(\eta * \chi)) \leq (\exp(c\eta)) * \chi$$

(by convexity of ‘exp’), therefore

$$\mathbb{E} \exp(c(\eta * \chi)) \leq (\mathbb{E} \exp(c\eta)) * \chi \stackrel{(5.2)}{\leq} C .$$

Thus,

$$\mathbb{P} ((\eta * \chi)(x) > c_3 \lambda^2 + c_3 |x|^2) \leq C \exp(-c_4 \lambda^2) \exp(-c_4 |x|^2) ,$$

and the sum over $x \in c_2 \mathbb{Z}^d$ does not exceed $C_1 \exp(-c_4 \lambda^2)$, which proves (5.3). \square

Formula (0.1) defines a Gaussian random process $\psi: \mathbb{C} \times \Omega_1 \rightarrow \mathbb{C}$ over the probability space $(\Omega_1, P_1) = (\mathbb{C}, \mathcal{N}_{\mathbb{C}}(0, 1))^{\{0,1,2,\dots\}}$ (the space of coefficients ζ_k). The random variable $\psi(0) = \zeta_0$ is distributed $\mathcal{N}_{\mathbb{C}}(0, 1)$. A simple exercise in integration shows that

$$\mathbb{P}(|\log |\zeta_0|| \geq s) = \mathbb{P}(|\zeta_0| \geq e^s) + \mathbb{P}(|\zeta_0| \leq e^{-s}) \leq e^{-e^{2s}} + e^{-2s} \leq 2e^{-2s},$$

therefore $\mathbb{E} \exp(|\log |\psi(0)||) < \infty$. We introduce a process $\varphi: \mathbb{C} \times \Omega_1 \rightarrow \mathbb{R}$ by (4.2). The distribution of the random potential $\varphi(z)$ does not depend on z (see [11]), therefore

$$\mathbb{E} \exp(2|\varphi(z)|) \leq \text{Const}.$$

By Lemma 5.1 applied to $\eta(z) = |\varphi(z)|$,

$$\mathbb{E} \exp(\text{const} \cdot R^2(z)) \leq \text{Const},$$

where $R(\cdot)$ is a random process defined by (4.1). It follows that $R(z) = O(\sqrt{\log |z|})$ for $|z| \rightarrow \infty$, almost surely. (The first part of the Borel-Cantelli lemma gives it for z on a lattice; the Lipschitz property of R extends it to all z .) By Theorem 4.3, for almost every $\omega \in \Omega_1$ there exists a two-dimensional array $(\xi_{k,l}(\omega))_{k,l \in \mathbb{Z}}$ of complex numbers such that

$$|\xi_{k,l}(\omega)| \leq R(\sqrt{\pi}(k + li))$$

for all k, l , and the set $\{\sqrt{\pi}(k + li) + \xi_{k,l}: k, l \in \mathbb{Z}\}$ is equal to the set $\psi^{-1}(0)$ of zeroes of ψ . However, the array need not be unique.

6 Final technicalities

In order to get *measurable* functions $\omega \mapsto \xi_{k,l}(\omega)$ one can use one of several well-known results about measurable selectors, such as [13, 5.2.1, 5.2.5, 5.2.6, 5.4.3, 5.5.8, 5.7.1, 5.12.1]. However, striving to keep the presentation reasonably elementary, we borrow from [13] only a special case of Corollary 5.2.4, formulated below.

Given a complete separable metric space X , we equip the set $K(X)$ of all compact subsets $K \subset X$ with the Hausdorff metric

$$\text{dist}(K_1, K_2) = \inf\{\delta > 0: K_1 \subset (K_2)_{+\delta}, K_2 \subset (K_1)_{+\delta}\}$$

and the corresponding Borel σ -field (generated by open subsets of $K(X)$). See the item ‘Spaces of compact sets’ in [13, Sect. 2.4], see also the item

‘Effros Borel space’ in [13, Sect. 3.3]. The topology of $K(X)$ (known as the Vietoris topology) is generated by sets of the following two forms:

$$\begin{aligned} &\{K \in K(X) : K \subset U\}, \\ &\{K \in K(X) : K \cap U \neq \emptyset\}, \end{aligned}$$

where U runs over open subsets of X . Unlike [13], we treat the empty set as a point of $K(X)$; it is an isolated point (take $U = \emptyset$ in the first form above). The reader can check that the same topology on $K(X)$ is generated also by functions $K(X) \rightarrow [-\infty, \infty)$ of the form

$$K \mapsto \max_K \varphi,$$

where φ runs over bounded continuous functions $X \rightarrow \mathbb{R}$; for $K = \emptyset$ the maximum is $-\infty$.

6.1 Theorem. *There exists a Borel map $s : K(X) \setminus \{\emptyset\} \rightarrow X$ such that $s(K) \in K$ for all nonempty $K \in K(X)$.*

A proof is given in [13, 5.2.4], but here is a hint: given ε , choose a countable ε -net $\{x_1, x_2, \dots\}$ of X and construct a Borel map $s_\varepsilon : K(X) \setminus \{\emptyset\} \rightarrow \{x_1, x_2, \dots\}$ such that $\text{dist}(s_\varepsilon(K), K) \leq \varepsilon$ for all K .

6.2 Lemma. *Let X be a complete separable metric space, Y a metric space, and $f : X \rightarrow Y$ a continuous map. Then the map $K(X) \times Y \rightarrow K(X)$ defined by*

$$(K, y) \mapsto K \cap f^{-1}(\{y\})$$

is a Borel map.

Proof. Let $\varphi : X \rightarrow \mathbb{R}$ be a bounded continuous function; it is sufficient to prove that the maximum of φ on $K \cap f^{-1}(\{y\})$ is a Borel function of (K, y) . We use penalization:

$$\max_{x \in K \cap f^{-1}(\{y\})} \varphi(x) = \lim_{n \rightarrow \infty} \max_{x \in K} (\varphi(x) - n \min(1, \text{dist}(f(x), y))).$$

For each n the expression is continuous in y uniformly in K and continuous in K for every y , therefore it is continuous in (K, y) . The limit of a pointwise convergent sequence of such functions is a Borel function. \square

We apply the lemma to the separable Banach space X of all two-dimensional arrays $(\xi_{k,l})$ of complex numbers satisfying $|\xi_{k,l}| = o(|k + li|)$, with the norm

$$\sup_{k,l} \frac{|\xi_{k,l}|}{1 + |k + li|}$$

(many other spaces could be used as well), the metrizable space Y of all locally finite measures on \mathbb{C} , equipped with the topology of local weak convergence, and the continuous map $f: X \rightarrow Y$ that sends $(\xi_{k,l})$ into the sum of unit-mass atoms at points $\sqrt{\pi}(k+li) + \xi_{k,l}$. Every function $R: \sqrt{\pi}\mathbb{Z}^2 \rightarrow [0, \infty)$ such that $R(z) = o(|z|)$ for $|z| \rightarrow \infty$ leads to a compact set $K_R \subset X$,

$$(\xi_{k,l}) \in K_R \iff \forall k, l \quad |\xi_{k,l}| \leq R(\sqrt{\pi}(k+li)).$$

The map $R \mapsto K_R$ is continuous w.r.t. the norm $\|R\| = \sup(R(z)/(1+|z|))$.

As was shown in Sect. 5, almost each $\omega \in \Omega_1$ leads to an entire function ψ_ω , a measure $n_\omega = n_{\psi_\omega}$ (recall (0.3)), and a function R_ω satisfying (much more than) $R_\omega(z) = o(|z|)$ and such that the compact set

$$K_\omega = K_{R_\omega} \cap f^{-1}(\{n_\omega\})$$

is nonempty. Measurability in ω of f_ω implies that of n_ω , R_ω and, by Lemma 6.2, of K_ω . Combined with Theorem 6.1, it gives us the following result.

6.3 Lemma. *There exist random variables $\xi_{k,l}: \Omega_1 \rightarrow \mathbb{C}$ such that*

$$(6.4) \quad \mathbb{E} \exp(\text{const} \cdot |\xi_{k,l}|^2) \leq \text{Const}$$

and for almost all ω the set $\{\sqrt{\pi}(k+li) + \xi_{k,l}: k, l \in \mathbb{Z}\}$ is equal to the set $\psi^{-1}(0)$ of zeroes of ψ .

The joint distribution of random variables $\xi_{k,l}$ is a probability measure on the space $\Omega = \mathbb{C}^{\mathbb{Z}^2}$ satisfying Items (a), (b) of the main theorem. However, the measure need not be shift-invariant.

The set of all probability measures on Ω satisfying both Item (a) of the main theorem and (6.4) is convex, weakly compact, and invariant under the action of \mathbb{Z}^2 (by continuous operators of shift). By the Markov-Kakutani theorem [4, Sect. 5.10.6], the action has a fixed point P in the set. This completes the proof of the main theorem.

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